

# Relative contravariantly finite subcategories and relative tilting modules <sup>★</sup>

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## Abstract

Let  $A$  be a finite dimensional algebra over an algebraically closed field  $k$ . Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A T$  be the endomorphism algebra of  $T$ . In this paper, we consider the correspondence between the tilting  $A$ -modules and the tilting  $B$ -modules, and we prove that there is a one-one correspondence between the basic  $T$ -tilting  $A$ -modules in  $T^\perp$  and the basic tilting  $B$ -modules in  ${}^\perp(D_B T)$ . Moreover, we show that there is a one-one correspondence between the  $T$ -contravariantly finite  $T$ -resolving subcategories of  $T^\perp$  and the basic  $T$ -tilting  $A$ -modules contained in  $T^\perp$ . As an application, we show that there is a one-one correspondence between the basic tilting  $A$ -modules in  $T^\perp$  and the basic tilting  $B$ -modules in  ${}^\perp(D_B T)$  if  $A$  is a 1-Gorenstein algebra or a  $m$ -replicated algebra over a finite dimensional hereditary algebra.

**Key words and phrases:** Right orthogonal category of tilting module; contravariantly finite subcategory;  $T$ -tilting module;  $T$ -resolving subcategory.

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# 1 Introduction

Let  $A$  be a finite-dimensional algebra over an algebraically closed field  $k$ . We denote by  $\text{mod-}A$  the category of all finitely generated right  $A$ -modules, and we always assume that subcategories of  $A$ -modules are closed under isomorphisms and direct summands.

Let  $\mathcal{M}$  be a subcategory of  $\text{mod-}A$ . We denote by  $\widehat{\mathcal{M}}$  the subcategory of  $\text{mod-}A$  consisting of the  $A$ -modules  $L$  such that there is an exact sequence  $0 \rightarrow M_n \rightarrow M_{n-1} \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow L \rightarrow 0$  with  $M_i \in \mathcal{M}$ , and we define  $\dim_{\widehat{\mathcal{M}}}(L)$  is the minimal  $n$  such that there exists an exact sequence  $0 \rightarrow M_n \rightarrow M_{n-1} \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow L \rightarrow 0$  with  $M_i \in \mathcal{M}$ . Dually, we can define  $\widetilde{\mathcal{M}}$  and  $\dim_{\widetilde{\mathcal{M}}}(L)$ .

For any  $0 < i < \infty$ , the following definition is taken from [6],  $\mathcal{M}^{\perp i} = \{X \in \text{mod-}A \mid \text{Ext}_A^i(M, X) = 0, \forall M \in \mathcal{M}\}$ , the right orthogonal category of  $\mathcal{M}$  is  $\mathcal{M}^{\perp} = \bigcap_{0 < i < \infty} \mathcal{M}^{\perp i} = \{X \in \text{mod-}A \mid \text{Ext}_A^i(M, X) = 0, \forall M \in \mathcal{M}, \forall i > 0\}$ . Dually,  ${}^{\perp i}\mathcal{M} = \{X \in \text{mod-}A \mid \text{Ext}_A^i(X, M) = 0, \forall M \in \mathcal{M}\}$ , the left orthogonal category of  $\mathcal{M}$  is  ${}^{\perp}\mathcal{M} = \bigcap_{0 < i < \infty} {}^{\perp i}\mathcal{M} = \{X \in \text{mod-}A \mid \text{Ext}_A^i(X, M) = 0, \forall i > 0, \forall M \in \mathcal{M}\}$ . In particular, when  $\mathcal{M} = \text{add } M$ , we just denote them by  $M^{\perp i}, M^{\perp}, {}^{\perp i}M, {}^{\perp}M$  respectively.

Let  $T$  be a tilting  $A$ -module. Denote by  $\mathcal{Y}_T$  the subcategory of  $\text{mod-}A$  whose objects are the  $A$ -modules  $Y$  in  $T^{\perp}$  for which there is an exact sequence

$$\cdots \xrightarrow{f_m} T_m \xrightarrow{f_{m-1}} T_{m-1} \xrightarrow{f_{m-2}} \cdots \xrightarrow{f_1} T_1 \xrightarrow{f_0} T_0 \longrightarrow Y \longrightarrow 0$$

with  $T_i$  in  $\text{add } T$  and  $\text{Ker } f_i$  in  $T^{\perp}$ . According to [1], we know that  $T^{\perp} = \mathcal{Y}_T$ .

Let  $M$  be an  $A$ -module in  $\mathcal{Y}_T$ . We denote by  $\dim_{\widehat{\text{add } T}}(M)$  ( $T$ -pd  $M$  for short) the  $T$ -projective dimension of  $M$ , and if  $M \in \text{add } T$ , then  $M$  is said to be  $T$ -projective. It follows that the subcategory  $\mathcal{T}^{<\infty}(T) = \widehat{\text{add } T}$  consists of all  $A$ -modules with finite  $T$ -projective dimensions. It is easy to see that  $\mathcal{T}^{<\infty}(T)$  is the subcategory of  $T^{\perp}$ . In particular, if  $T = A$ , we have  $\mathcal{T}^{<\infty}(T) = \mathcal{P}^{<\infty}(\text{mod-}A)$ .

Tilting theory is a central topic in the representation theory of algebras, which has two aspects. One is the external aspect, which is usually used to compare  $\text{mod-}A$  to  $\text{mod-End}_A T$  for a tilting  $A$ -module  $T$ , and the other is the internal aspect, which is to study the structure properties of tilting modules for a fixed algebra  $A$ .

Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A T$ . Recall from [13], Miyashita proved some correspondence between the orthogonal subcategories of  $\text{mod-}A$  and  $\text{mod-}B$ . In [1], Auslander and Reiten have proved that there is a one-one correspondence between the basic tilting  $A$ -modules and the contravariantly finite resolving subcategories of  $\text{mod-}A$ . However, the relationship between the tilting  $A$ -modules and the tilting  $B$ -modules is little known. In this paper, we focus on the investigation on the tilting modules in the orthogonal subcategories and show that there exists a one-one correspondence between the basic  $T$ -tilting  $A$ -modules (defined in next section) in  $T^\perp$  and basic tilting  $B$ -modules in  ${}^\perp(D_B T)$ . As an application, we will prove that there is a one-one correspondence between the basic tilting  $A$ -modules in  $T^\perp$  and the basic tilting  $B$ -modules in  ${}^\perp(D_B T)$  if  $A$  is a 1-Gorenstein algebra or a  $m$ -replicated algebra over a finite dimensional hereditary algebra.

Now, we state our main results in this paper.

**Theorem 1** *Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . Then  $\text{Hom}_A(T, -)$  and  $-\otimes_B T$  give a one-one correspondence between isomorphism classes of basic  $T$ -tilting  $A$ -modules in  $T^\perp$  and of basic tilting  $B$ -modules in the subcategory  ${}^\perp(D_B T)$  of  $\text{mod-}B$ .*

We should mention that it is not sure whether a  $T$ -tilting  $A$ -module is tilting, but for 1-Gorenstein algebras and  $m$ -replicated algebras over a finite dimensional hereditary algebras, this is true.

**Theorem 2** *Let  $A$  be a 1-Gorenstein algebra or a  $m$ -replicated algebra over a finite dimensional hereditary algebra, and let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . Then there is a one-one correspondence between the isomorphism classes of basic tilting  $A$ -modules in  $T^\perp$  and of basic tilting  $B$ -modules in the subcategory  ${}^\perp(D_B T)$  of  $\text{mod-}B$ .*

In general, a  $T$ -tilting  $A$ -module is partial tilting. Moreover, we have

**Theorem 3** *Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . Then  $\text{Hom}_A(T, -)$  and  $- \otimes_B T$  give a one-one correspondence between basic partial tilting  $A$ -modules in  $T^\perp$  and basic partial tilting  $B$ -modules in  ${}^\perp(D_B T)$ .*

The conceptions of contravariantly and covariantly finite subcategories of  $\text{mod } A$  were introduced in [3, 4] by Auslander and Smalø when they studied the problem of which subcategories of  $\text{mod } A$  have almost split sequence. These conceptions have close relationships with tilting modules and cotilting modules, see [1] for details.

Let  $\mathcal{C}$  be a full subcategory of  $\text{mod } A$ ,  $C_M \in \mathcal{C}$  and  $\varphi : C_M \longrightarrow M$  with  $M \in A\text{-mod}$ . The morphism  $\varphi$  is a right  $\mathcal{C}$ -approximation of  $M$  if the induced morphism  $\text{Hom}_A(C, C_M) \longrightarrow \text{Hom}_A(C, M)$  is surjective for any  $C \in \mathcal{C}$ . A minimal right  $\mathcal{C}$ -approximation of  $M$  is a right  $\mathcal{C}$ -approximation which is also a right minimal morphism, i.e., its restriction to any nonzero summand is nonzero. The subcategory  $\mathcal{C}$  is called contravariantly finite if any module  $M \in A\text{-mod}$  admits a (minimal) right  $\mathcal{C}$ -approximation. The notions of (minimal) left  $\mathcal{C}$ -approximation and of covariantly finite subcategory are dually defined. It is well known that  $\text{add } M$  is both a contravariantly finite subcategory and a covariantly finite subcategory.

Let  $\mathcal{X}$  be a subcategory of  $\text{mod } A$ .  $\mathcal{X}$  is said to be a resolving (resp. coresolving) subcategory if it is closed under extensions, the kernels of epimorphisms (resp. the cokernels of monomorphisms) and contains all indecomposable projective (resp. injective)  $A$ -modules. Auslander and Reiten have proved in [1] that there is a one-one correspondence between the basic tilting  $A$ -modules and the contravariantly finite resolving subcategories of  $\text{mod } A$ .

In this paper, we generalize these kinds of correspondences, and show some one-one correspondence between the  $T$ -contravariantly finite  $T$ -resolving subcategories of  $T^\perp$  and  $T$ -tilting  $A$ -modules contained in  $T^\perp$ .

**Theorem 4** *Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A T$ . Then there is a one-one correspondence between the  $T$ -resolving  $T$ -contravariantly finite subcategories*

of  $T^\perp$  and the contravariantly finite resolving subcategories in  ${}^\perp(D_B T)$ .

Let  $T$  be a tilting  $A$ -module, and let  $L$  and  $L'$  be  $T$ -tilting modules. We say  $L$  and  $L'$  are equivalent if  $\text{add}L = \text{add}L'$ .

**Theorem 5** *Let  $T$  be a tilting  $A$ -module. Then for any  $L \in T^\perp$ ,  $\text{add}L \rightarrow \widetilde{\text{add}L} \cap T^\perp$  and  $\mathcal{U} \rightarrow \mathcal{U} \cap \mathcal{U}^\perp$  give a one-one correspondence between equivalence class of basic  $T$ -tilting  $A$ -modules in  $T^\perp$  and the  $T$ -contravariantly finite  $T$ -resolving subcategories  $\mathcal{U}$  of  $T^\perp$  which is contained in  $\mathcal{T}^{<\infty}(T)$ .*

For some special kinds of algebras, we have the following result, which seems to have independent interests.

**Theorem 6** *Let  $A$  be either a 1-Gorenstein algebra or a  $m$ -replicated algebra over a hereditary algebra, and  $T$  be a tilting  $A$ -module. Then there is a one-one correspondence between the equivalence classes of basic tilting  $A$ -modules in  $T^\perp$  and the  $T$ -contravariantly finite  $T$ -resolving subcategories contained in  $\mathcal{T}^{<\infty}(T)$ .*

This paper is arranged as the following. In section 2, we fix the notations and recall some necessary facts needed for our further research. Section 3 is devoted to the proof of Theorem 1, Theorem 2 and Theorem 3. In section 4, we prove Theorem 4, Theorem 5 and Theorem 6.

## 2 Preliminary

Let  $A$  be a finite dimensional algebra over an algebraically closed field  $k$ . We denote by  $\text{mod-}A$  the category of all finitely generated right  $A$ -modules and by  $\text{ind-}A$  the full subcategory of  $\text{mod-}A$  containing exactly one representative of each isomorphism class of indecomposable  $A$ -modules.  $D = \text{Hom}_k(-, k)$  is the standard duality between  $A\text{-mod}$  and  $A^{op}\text{-mod}$ , and  $\tau_A$  is the Auslander-Reiten translation of  $A$ . We denote by  $\text{gl.dim } A$  the global dimension of  $A$ . The Auslander-Reiten quiver of  $A$  is denoted by  $\Gamma_A$ .

Given an  $A$ -module  $M$ , we denote by  $\text{pd}M$  the projective dimension of  $M$  and

by add  $M$  the full subcategory having as objects the direct sums of indecomposable summands of  $M$ . For a subcategory  $\mathcal{M}$  of  $\text{mod-}A$ , we denote by  $\text{add}\mathcal{M}$  the subcategory of  $\text{mod-}A$  consisting of all direct summand of finitely indecomposable modules in  $\mathcal{M}$ .

Given any module  $M \in A\text{-mod}$ , we may decompose  $M$  as  $M \cong \bigoplus_{i=1}^m M_i^{d_i}$ , where each  $M_i$  is indecomposable,  $d_i > 0$  for each  $i$ , and  $M_i$  is not isomorphic to  $M_j$  if  $i \neq j$ . The module  $M$  is called basic if  $d_i = 1$  for any  $i$ . The number of non-isomorphic indecomposable modules occurring in the direct sum decomposition above is uniquely determined and it is denoted by  $\delta(M)$ .

An  $A$ -module  $T$  in  $\text{mod-}A$  is called a (generalized) tilting module if the following conditions are satisfied:

- (1)  $\text{pd}T = n < \infty$ ;
- (2)  $\text{Ext}_A^i(T, T) = 0$  for all  $i > 0$ ;
- (3) There is a long exact sequence

$$0 \longrightarrow A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_n \longrightarrow 0$$

with  $T_i \in \text{add } T$  for  $0 \leq i \leq n$ .

An  $A$ -module  $M$  satisfying the conditions (1) and (2) of the definition above is called a partial tilting module. Let  $M$  be a partial tilting module and  $X$  be an  $A$ -module such that  $M \oplus X$  is a tilting module and  $\text{add}M \cap \text{add}X = 0$ . Then  $X$  will be called a complement to  $M$ .

It is well known that in the classical situation  $M$  always admits a complement and  $M$  is a tilting module if and only if  $\delta(M) = \delta(A)$ . However, in general situations complements do not always exist, as shown in [15]. Moreover it is an important open problem whether  $\delta(M) = \delta(A)$  is sufficient for a partial tilting module  $M$  to be a tilting module.

**Definition 2.1** *Let  $\mathcal{U} \subseteq \mathcal{V} \subseteq \text{mod}A$ . Then we have:*

- (1) *If for any  $V \in \mathcal{V}$ , there is a right  $\mathcal{U}$ -approximation of  $V$ , then we call  $\mathcal{U}$  is*

contravariantly finite in  $\mathcal{V}$ .

(2) If for any  $V \in \mathcal{V}$ , there is a left  $\mathcal{U}$ -approximation of  $V$ , then we call  $\mathcal{U}$  is covariantly finite in  $\mathcal{V}$ .

(3) If  $\mathcal{U}$  are both contravariantly finite and covariantly finite in  $\mathcal{V}$ , we call  $\mathcal{U}$  is functorially finite in  $\mathcal{V}$ .

Let  $T$  be a tilting  $A$ -module. If a subcategory  $\mathcal{U}$  of  $T^\perp$  is contravariantly finite in  $T^\perp$ , then  $\mathcal{U}$  is said to be  $T$ -contravariantly finite. Now, we define  $T$ -resolving subcategory as following.

**Definition 2.2** Let  $\mathcal{U} \subseteq T^\perp$ . Then  $\mathcal{U}$  is said to be a  $T$ -resolving subcategory, if it satisfies the following conditions:

(1)  $\mathcal{U}$  is closed under extensions;

(2)  $\mathcal{U}$  is closed under the kernels of epimorphisms in  $T^\perp$ , i.e., for a short exact sequence  $0 \rightarrow U_1 \rightarrow U_2 \rightarrow U_3 \rightarrow 0$  in  $T^\perp$ , if  $U_2, U_3 \in \mathcal{U}$ , then  $U_1 \in \mathcal{U}$ .

(3)  $\text{add}T \subseteq \mathcal{U}$ .

We can define  $T$ -coresolving subcategory similarly.

**Lemma 2.1** Let  $\mathcal{U}$ ,  $\mathcal{V}$  and  $\mathcal{W}$  be subcategories of  $\text{mod-}A$  with  $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{W}$ . Then we have:

(1) If  $\mathcal{U}$  is contravariantly finite in  $\mathcal{V}$  and  $\mathcal{V}$  is contravariantly finite in  $\mathcal{W}$ , then  $\mathcal{U}$  is contravariantly finite in  $\mathcal{W}$ .

(2) If  $\mathcal{U}$  is covariantly finite in  $\mathcal{V}$  and  $\mathcal{V}$  is covariantly finite in  $\mathcal{W}$ , then  $\mathcal{U}$  is covariantly finite in  $\mathcal{W}$ .

**Lemma 2.2** [1, Proposition 3.7] Let  $\mathcal{U}$  be a resolving subcategory. Then:

(1) The subcategory of all modules that have right  $\mathcal{U}$ -approximation is closed under extensions.

(2)  $\mathcal{U}$  is contravariantly finite in  $\text{mod}A$  if and only if all simple right  $A$  modules

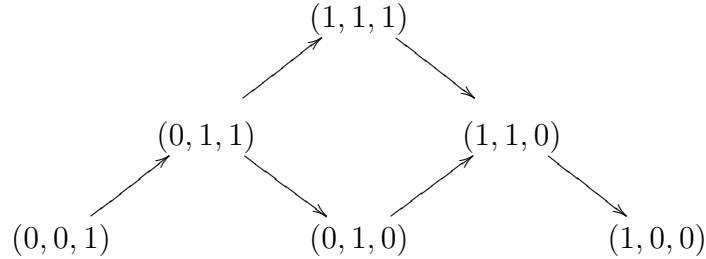
have right  $\mathcal{U}$ -approximations.

Let  $T$  be a tilting module in  $\text{mod-}A$ , we now define a special partial tilting module, called  $T$ -tilting module.

**Definition 2.3** *Let  $T$  be a tilting module in  $\text{mod-}A$  and  $L \in T^\perp$ .  $L$  is said to be a  $T$ -tilting module if it satisfies the following conditions:*

- (1)  $\text{T-pd } L = n < \infty$ ;
- (2)  $\text{Ext}_A^i(L, L) = 0$  for  $0 < i < \infty$ ;
- (3) *There exists an exact sequence  $0 \rightarrow T \rightarrow L_0 \rightarrow L_1 \cdots \rightarrow L_m \rightarrow 0$  with  $L_i \in \text{add } L$ .*

**Example** Let  $A = kQ$  be the path algebra of  $Q$  with  $Q : 1 \rightarrow 2 \rightarrow 3$ . The AR quiver  $\Gamma_A$  of  $A$  is:



We denote by  $P_1 = (1, 1, 1)$ ,  $P_2 = (0, 1, 1)$ ,  $T_1 = (1, 1, 0)$  and  $T_2 = (0, 1, 0)$ . Then  $T = P_1 \oplus P_2 \oplus T_2$  is a tilting  $A$ -module. Let  $T' = P_1 \oplus T_1 \oplus T_2$ . Then we have an exact sequence  $0 \rightarrow T \rightarrow P_1 \oplus P_1 \oplus T_2 \oplus T_2 \rightarrow T_1 \rightarrow 0$ ,  $P_1 \oplus P_1 \oplus T_2 \oplus T_2 \in \text{add } T'$ , and  $T'$  is a  $T$ -tilting module.

Obviously, the  $T$ -tilting module in the above example is a tilting module, in particular,  $T$ -tilting modules over hereditary algebras are tilting modules. In general, all  $T$ -tilting modules are partial tilting modules.

**Proposition 2.3** *Let  $T$  be a tilting  $A$ -module and  $L$  be a  $T$ -tilting  $A$ -module,*



then  $L$  is a partial tilting  $A$ -module.

**Proof** Note that  $\text{Ext}_A^i(L, L) = 0$  for  $i > 0$  since  $L$  is a  $T$ -tilting module. We only need to show  $\text{pd}_A L < \infty$ .

Since  $T\text{-pd} L = m < \infty$ , there is an exact sequence  $0 \rightarrow T_m \xrightarrow{f_m} T_{m-1} \cdots \xrightarrow{f_1} T_0 \xrightarrow{f_0} L \rightarrow 0$  with  $T_i \in \text{add} T$ .

Let  $C_i = \text{cokernel}(f_i)$ . Then  $L = C_1$ , and we have following exact sequence:

$$0 \longrightarrow T_m \longrightarrow T_{m-1} \longrightarrow C_m \longrightarrow 0$$

$$0 \longrightarrow C_{i+1} \longrightarrow T_{i-1} \longrightarrow C_i \longrightarrow 0 (0 < i < m).$$

$T$  is a tilting module, hence  $\text{pd}_A T \leq n < \infty$ , and so we get  $\text{pd}_A C_m \leq n + 1$ ,  $\text{pd}_A C_{m-1} \leq n + 2, \dots, \text{pd}_A L = \text{pd}_A C_1 \leq n + m < \infty$ . Hence  $L$  is a partial tilting  $A$ -module.  $\square$

The following lemmas are well known and useful in our research.

**Lemma 2.4** [1, Proposition 3.4 (c)] *Let  $T$  be a tilting module in  $\text{mod-}A$ . Then for any  $M \in T^\perp$ , there is an exact sequence  $0 \rightarrow K \rightarrow T' \rightarrow M \rightarrow 0$  with  $T' \in \text{add} T$  and  $K \in T^\perp$ .*

**Remark.** Let  $T$  be a tilting  $A$ -module. For any  $M \in T^\perp$ , there exists a long exact sequence  $\cdots \rightarrow T_2 \rightarrow T_1 \rightarrow T_0 \rightarrow M \rightarrow 0$  with  $T_i \in \text{add} T$ .

**Lemma 2.5** [13, Proposition 1.20] *Let  $T$  be a tilting  $A$ -module. Then for any  $M, N \in T^\perp$ , we have:*

$$(1) \text{Hom}_B(\text{Hom}_A(T, M), \text{Hom}_A(T, N)) \cong \text{Hom}_A(M, N).$$

$$(2) \text{Ext}_B^j(\text{Hom}_A(T, M), \text{Hom}_A(T, N)) \cong \text{Ext}_A^j(M, N), 0 < j < \infty.$$

**Lemma 2.6** [13, Theorem 1.16] *Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . Then  $\text{Hom}_A(T, -)$  and  $-\otimes_B T$  give a one-one correspondence between the indecomposable modules in the subcategory  $T^\perp$  of  $\text{mod-}A$  and the indecomposable modules in the subcategory  ${}^\perp(D_B T)$  of  $\text{mod-}B$ .*

According to [1], we know that there is close relationship between tilting modules and contravariantly finite (or covariantly finite) subcategories. Two tilting  $A$ -modules  $T$  and  $T'$  are said to be equivalent if  $\text{add}T = \text{add}T'$ . The following lemma is taken from [1].

**Lemma 2.7** [1, Theorem 5.5] *Let  $T$  be an  $A$ -module. Then we have:*

(1)  $T \rightarrow T^\perp$  gives a one-one correspondence between equivalence class of basic tilting  $A$ -modules and covariantly finite coresolving subcategories  $\mathcal{Y}$  with  $\check{\mathcal{Y}} = \text{mod-}A$ , and the inverse correspondence is given by  $\mathcal{Y} \rightarrow {}^\perp\mathcal{Y} \cap \mathcal{Y}$ .

(2)  $T \rightarrow \widetilde{\text{add}T}$  gives a one-one correspondence between equivalence class of basic tilting modules and contravariantly finite resolving subcategories  $\mathcal{X}$  with  $\mathcal{X} \subseteq \mathcal{P}^{<\infty}(\text{mod}A)$ , and the inverse correspondence is given by  $\mathcal{X} \rightarrow \mathcal{X} \cap \mathcal{X}^\perp$ .

(3)  $T \rightarrow {}^\perp T$  gives a one-one correspondence between equivalence class of basic cotilting modules and contravariantly finite resolving subcategories  $\mathcal{X}$  with  $\hat{\mathcal{X}} = \text{mod-}A$ , and the inverse correspondence is given by  $\mathcal{X} \rightarrow \mathcal{X} \cap \mathcal{X}^\perp$ .

(4)  $T \rightarrow \widehat{\text{add}T}$  gives a one-one correspondence between equivalence class of basic cotilting modules and covariantly finite coresolving subcategories  $\mathcal{Y}$  with  $\mathcal{Y} \subseteq \mathcal{I}^{<\infty}(\text{mod}A) = \{M \in \text{mod-}A \mid \text{id}_A M < \infty\}$ , and the inverse correspondence is given by  $\mathcal{Y} \rightarrow \mathcal{Y} \cap \mathcal{Y}^\perp$ .

Throughout this paper, we follow the standard terminology and notation used in the representation theory of algebras, see [2, 5]

### 3 $T$ -tilting modules

Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . According to Lemma 2.6,  $\text{Hom}_A(T, -)$  and  $- \otimes_B T$  induce a one-one correspondence between the modules in  $T^\perp$  and the modules in  ${}^\perp(D_B T)$ . In this section, we shall show that there exists a one-one correspondence between the basic  $T$ -tilting  $A$ -modules in  $T^\perp$  and the basic tilting  $B$ -modules in  ${}^\perp(D_B T)$ . Moreover,  $\text{Hom}_A(T, -)$  and  $- \otimes_B T$  also give

a one-one correspondence between the basic partial tilting  $A$ -modules in  $T^\perp$  and the basic partial tilting  $B$ -modules in  ${}^\perp(D_B T)$ .

**Lemma 3.1** *Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . Let  $M \in T^\perp$ . Then  $T\text{-pd}M < \infty$  if and only if  $\text{pd}_B(\text{Hom}_A(T, M)) < \infty$ . In fact, we have  $T\text{-pd}M = \text{pd}_B \text{Hom}_A(T, M)$ .*

**Proof** Let  $M \in T^\perp$  and  $T\text{-pd}M = k < \infty$ . Then there is a long exact sequence:

$$0 \rightarrow T_k \xrightarrow{f_k} T_{k-1} \cdots \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \rightarrow 0,$$

with  $T_i \in \text{add}T \subseteq T^\perp$  and  $C_i = \text{cokernel}(f_i) \in T^\perp$ . Applying  $\text{Hom}_A(T, -)$  yields an exact sequence:

$$0 \rightarrow \text{Hom}_A(T, T_k) \rightarrow \text{Hom}_A(T, T_{k-1}) \cdots \rightarrow \text{Hom}_A(T, T_0) \rightarrow \text{Hom}_A(T, M) \rightarrow 0.$$

Note that  $T_i \in \text{add}T$ ,  $\text{Hom}_A(T, T) = B$ , hence  $\text{Hom}_A(T, T_i)$  are projective  $B$ -modules, and  $\text{pd}_B(\text{Hom}_A(T, M)) \leq k < \infty$ .

On the other hand, we assume that  $\text{pd}_B(\text{Hom}_A(T, M)) = k < \infty$ . Then there is a projective resolution of  $\text{Hom}_A(T, M)$  in  $\text{mod-}B$ :

$$(*) \quad 0 \rightarrow P_k \xrightarrow{g_k} P_{k-1} \cdots \xrightarrow{g_1} P_0 \xrightarrow{g_0} \text{Hom}_A(T, M) \rightarrow 0, P_i \in \text{add}B.$$

Note that  $\text{Hom}_A(T, M) \in {}^\perp(D_B T)$ , and  $K_i = \ker g_i \in {}^\perp(D_B T)$ . Hence we have

$$D\text{Tor}_1^B(K_{i,B} T) \cong \text{Ext}_B^1(K_i, {}^\perp(D_B T)) = 0.$$

Applying  $-\otimes_B T$  to  $(*)$  yields an exact sequence:

$$0 \rightarrow P_k \otimes_B T \rightarrow P_{k-1} \otimes_B T \cdots \rightarrow P_0 \otimes_B T \rightarrow \text{Hom}_A(T, M) \otimes_B T \rightarrow 0.$$

Since  $P_i \otimes_B T \in \text{add}T$  and  $\text{Hom}_A(T, M) \otimes_B T \cong M$ , we have  $T\text{-pd}M \leq k < \infty$ .

Following from the above proof, we obtain that  $T\text{-pd}M$  is infinite if and only if  $\text{pd}_B(\text{Hom}_A(T, M))$  is infinite. The proof is completed.  $\square$

Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . According to Lemma 2.6,  $\text{Hom}_A(T, -)$  and  $-\otimes_B T$  give a one-one correspondence between  $A$ -modules in  $T^\perp$  and  $B$ -modules in  ${}^\perp(D_B T)$ . Moreover, we shall show that they also induce a one-one correspondence between  $T$ -tilting  $A$ -modules in  $T^\perp$  and tilting  $B$ -modules in the subcategory  ${}^\perp(D_B T)$  of  $\text{mod-}B$ .

**Theorem 3.2** *Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . Then  $\text{Hom}_A(T, -)$  and  $-\otimes_B T$  give a one-one correspondence between isomorphism classes of basic  $T$ -tilting  $A$ -modules in  $T^\perp$  and of basic tilting  $B$ -modules in the subcategory  ${}^\perp(D_B T)$  of  $\text{mod-}B$ .*

**Proof** Let  $L \in T^\perp$  be a basic  $T$ -tilting  $A$ -module. According to Lemma 2.6,  $\text{Hom}_A(T, L)$  is a basic  $B$ -module belonging to  ${}^\perp(D_B T)$ . By using Lemma 3.1, we know that

$$\text{pd}_B(\text{Hom}_A(T, L)) = T - \text{pd} L < \infty.$$

By Lemma 2.5(2), we know that  $\text{Ext}_B^j(\text{Hom}_A(T, L), \text{Hom}_A(T, L)) \cong \text{Ext}_A^j(L, L) = 0$ , for all  $j > 0$ . Since  $L$  is a  $T$ -tilting  $A$ -module, we have an exact sequence

$$0 \rightarrow T \rightarrow L_0 \rightarrow L_1 \cdots \rightarrow L_m \rightarrow 0 \text{ with } L_i \in \text{add } L.$$

Applying the functor  $\text{Hom}_A(T, -)$  yields an exact sequence

$$0 \rightarrow \text{Hom}_A(T, T) \rightarrow \text{Hom}_A(T, L_0) \rightarrow \text{Hom}_A(T, L_1) \cdots \rightarrow \text{Hom}_A(T, L_m) \rightarrow 0.$$

It follows that  $\text{Hom}_A(T, L)$  is a tilting  $B$ -module in  ${}^\perp(D_B T)$ .

On the other hand, let  $M \in {}^\perp(D_B T)$  be a basic tilting  $B$ -module. Then  $M \otimes_B T \in T^\perp$  and there exists an  $A$ -module  $L \in T^\perp$  such that  $M = \text{Hom}_A(T, L)$  by using Lemma 2.6.

Let  $\text{pd } M = m < \infty$ . Then there exists a minimal projective resolution of  $M$

$$0 \rightarrow P_m \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

with  $P_i = \text{Hom}_A(T, T_i)$  for some  $T_i \in \text{add } T$ . Note that  $\text{Tor}_i^B(M, T) = 0$  since  $D\text{Tor}_i^B(M, T) \simeq \text{Ext}_B^i(M, DT) = 0$  for all  $i > 0$ . Applying the functor  $-\otimes_B T$  to

the above sequence yields an exact sequence

$$0 \rightarrow T_m \rightarrow \cdots \rightarrow T_1 \rightarrow T_0 \rightarrow L \rightarrow 0.$$

That is  $T - \text{pd } L \leq m$ .

According to Lemma 2.5 (2),  $\text{Ext}_A^i(L, L) \simeq \text{Ext}_B^i(\text{Hom}_A(T, L), \text{Hom}_A(T, L)) = 0$ .

Finally, since we have an exact sequence

$$0 \rightarrow B \rightarrow M_1 \rightarrow M_2 \cdots \rightarrow M_s \rightarrow 0$$

with  $M_i \in \text{add } M$  and  $\text{Tor}_i^B(M, T) = 0$ , applying the functor  $- \otimes_B T$  yields an exact sequence

$$0 \rightarrow T \rightarrow L_1 \rightarrow L_2 \cdots \rightarrow L_s \rightarrow 0,$$

which means that  $M \otimes_B T = L$  is a  $T$ -tilting  $A$ -module. The proof is completed.

□

**Remark** Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . Let  $L$  be a basic  $T$ -tilting  $A$ -module in  $T^\perp$ . It follows that the number of indecomposable direct summands of  $L$  is equal to the number of simple  $A$ -modules. According to Proposition 2.3, we know that  $L$  is also a partial tilting  $A$ -module. In general, we are not sure whether a  $T$ -tilting  $A$ -module in  $T^\perp$  is a tilting  $A$ -module. However, we have the following corollary.

**Corollary 3.3** *Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . Then a  $T$ -tilting  $A$ -module is also a tilting  $A$ -module, if  $A$  is one of the following kinds of algebras:*

- (1)  *$A$  is a 1-Gorenstein algebra.*
- (2)  *$A$  is a  $m$ -replicated algebra over a finite dimensional hereditary algebra.*

**Proof** One can easily see that if every partial tilting  $A$ -module  $M$  with  $\delta(M) = \delta(A)$  is tilting, then every  $T$ -tilting  $A$ -module is also a tilting  $A$ -module.

- (1) If  $A$  is a 1-Gorenstein algebra, then every partial tilting  $A$ -module  $M$  has projective dimension at most 1.

(2) If  $A$  is a  $m$ -replicated algebra over a finite dimensional hereditary algebra, then according to Theorem 3.1 in [17], every partial tilting  $A$ -module  $M$  with  $\delta(M) = \delta(A)$  is tilting.  $\square$

**Lemma 3.4** *Let  $T$  be a tilting  $A$ -module and  $M \in T^\perp$ . Then  $\text{pd}_A(M) < \infty$  if and only if  $\text{T-pd } M < \infty$ .*

**Proof** Note that  $M \in T^\perp$ . If  $\text{T-pd } M < \infty$ , then by Lemma 2.3, we have  $\text{pd}_A(M) < \infty$ .

On the other hand, assume that  $\text{pd}_A(M) = n < \infty$ . According to Lemma 2.4,  $M \in T^\perp$  implies that there exists an exact sequence:

$$0 \rightarrow K_n \rightarrow T_n \xrightarrow{f_k} T_{n-1} \cdots \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \rightarrow 0, T_i \in \text{add } T \subseteq T^\perp$$

with  $K_i = \ker f_i \in T^\perp (0 \leq i \leq n)$ . It follows that

$$\text{Ext}_A^1(K_{n-1}, K_n) \simeq \text{Ext}_A^2(K_{n-2}, K_n) \simeq \cdots \simeq \text{Ext}_A^n(K_0, K_n) \simeq \text{Ext}_A^{n+1}(M, K_n) = 0$$

since  $\text{pd}_A(M) = n$ . Hence the short exact sequence  $0 \rightarrow K_n \rightarrow T_n \rightarrow K_{n-1} \rightarrow 0$  splits, and  $K_{n-1}$  is a direct summand of  $T_n$ , thus  $\text{T-pd } M \leq n < \infty$ . The proof is completed.  $\square$

By using Corollary 3.3 and Lemma 3.4, we have

**Theorem 3.5** *Let  $A$  be a 1-Gorenstein algebra or a  $m$ -replicated algebra over a finite dimensional hereditary algebra, and let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . Then there is a one-one correspondence between the isomorphism classes of basic tilting  $A$ -modules in  $T^\perp$  and of basic tilting  $B$ -modules in the subcategory  ${}^\perp(D_B T)$  of  $\text{mod } B$ .*

**Proof** We only need to show that  $T$ -tilting  $A$ -module coincide with tilting  $A$ -module in  $T^\perp$ . If  $L \in T^\perp$  is a  $T$ -tilting  $A$ -module, then  $L$  is a tilting  $A$ -module by using Corollary 3.3.

Conversely, assume that  $L \in T^\perp$  is a tilting  $A$ -module. By Lemma 3.4,  $\text{T-pd } L < \infty$  since  $\text{pd}_A L < \infty$ , and  $\text{Ext}_A^i(L, L) = 0$  for all  $i > 0$ . Note that whether

$A$  is a 1-Gorenstein algebra or a  $m$ -replicated algebra, a tilting  $A$ -module also is a cotilting module, by using Theorem 5.4 in [1], we know that there is an exact sequence

$$0 \rightarrow T \longrightarrow L_0 \xrightarrow{f_0} L_1 \xrightarrow{f_1} L_2 \cdots \xrightarrow{f_n} L_n \rightarrow 0, \text{ with } L_i \in \text{add} T \subseteq T^\perp.$$

Thus  $L$  is a  $T$ -tilting  $A$ -module. The proof is completed.  $\square$

In general, we have the following correspondence between partial tilting  $A$ -modules in  $T^\perp$  and partial tilting  $B$ -modules in  ${}^\perp(D_B T)$ .

**Theorem 3.6** *Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . Then  $\text{Hom}_A(T, -)$  and  $- \otimes_B T$  give a one-one correspondence between basic partial tilting  $A$ -modules in  $T^\perp$  and basic partial tilting  $B$ -modules in  ${}^\perp(D_B T)$ .*

**Proof** According to Lemma 2.6,  $\text{Hom}_A(T, -)$  and  $- \otimes_B T$  give a one-one correspondence between basic  $A$ -modules in  $T^\perp$  and basic  $B$ -modules in  ${}^\perp(D_B T)$ . By Lemma 2.5(2), for every  $A$ -module  $M$  in  $T^\perp$ , we know that  $\text{Ext}_A^i(M, M) = 0$  ( $0 < i < \infty$ ) if and only if  $\text{Ext}_B^i(\text{Hom}_A(T, M), \text{Hom}_A(T, M)) = 0$  ( $0 < i < \infty$ ). By using Lemma 3.1 and Lemma 3.4, for any  $M \in T^\perp$ , we know that  $\text{pd}_A(M) < \infty$  if and only if  $\text{pd}_B(\text{Hom}_A(T, M)) < \infty$ . Hence, for an  $A$ -module  $A$  in  $T^\perp$ ,  $M$  is partial tilting if and only if  $\text{Hom}_A(T, M)$  is a tilting  $B$ -module belonging to  ${}^\perp(D_B T)$ . This completes the proof.  $\square$

## 4 $T$ -contravariantly finite subcategories

Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . In this section, we prove that there is a one-one correspondence between basic  $T$ -tilting  $A$ -modules in  $T^\perp$  and the  $T$ -contravariantly finite  $T$ -resolving subcategories in  $\mathcal{T}^{<\infty}(T)$ .

**Lemma 4.1** *Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . Then  $\text{Hom}_A(T, -)$  and  $- \otimes_B T$  induce a one-one correspondence between the  $T$ -contravariantly finite subcategories of  $T^\perp$  and contravariantly finite subcategories of  ${}^\perp(D_B T)$ .*

**Proof** Let  $\mathcal{U}$  be a  $T$ -contravariantly finite subcategory in  $T^\perp$ . Then  $\text{Hom}_A(T, \mathcal{U}) = \{\text{Hom}_A(T, U) \mid U \in \mathcal{U}\}$  is a subcategory of  ${}^\perp(D_B T)$ .

For any  $Y = \text{Hom}_A(T, M) \in {}^\perp(D_B T)$  with  $M \in T^\perp$ , since  $\mathcal{U}$  is contravariantly finite in  $T^\perp$ , there exists an  $A$ -module  $U_M \in \mathcal{U}$  and  $f_M \in \text{Hom}_A(U_M, M)$  such that  $\text{Hom}_A(U, f_M) : \text{Hom}_A(U, U_M) \rightarrow \text{Hom}_A(U, M)$  is surjective for any  $U \in \mathcal{U}$ . Hence  $V_Y = \text{Hom}_A(T, U_M) \in \text{Hom}_A(T, \mathcal{U})$ ,  $g_Y = \text{Hom}_A(T, f_M) \in \text{Hom}_B(V_Y, Y)$ . For any  $V = \text{Hom}_A(T, U) \in \text{Hom}_A(T, \mathcal{U})$ , we have the following commutative diagram:

$$\begin{array}{ccc} \text{Hom}_B(\text{Hom}_A(T, U), \text{Hom}_A(T, U_M)) & \xrightarrow{t_1} & \text{Hom}_A(U, U_M) \\ \text{Hom}_B(V, g_Y) \downarrow & & \downarrow \text{Hom}_A(U, f_M) \\ \text{Hom}_B(\text{Hom}_A(T, U), \text{Hom}_A(T, M)) & \xrightarrow{t_2} & \text{Hom}_A(U, M) \end{array}$$

where  $t_1$  and  $t_2$  are isomorphisms.  $\text{Hom}_A(U, f_M)$  is surjective and this implies that  $\text{Hom}_B(V, g_Y)$  is surjective. Thus  $\text{Hom}_A(T, \mathcal{U})$  is contravariantly finite in  ${}^\perp(D_B T)$ .

Similarly, one can prove that if  $\mathcal{V}$  is a contravariantly finite subcategory in  ${}^\perp(D_B T)$ , then  $\mathcal{V} \otimes_B T$  is a  $T$ -contravariantly finite subcategory in  $T^\perp$ . This completes the proof.  $\square$

**Lemma 4.2** *Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A(T)$ . Then  $\text{Hom}_A(T, -)$  and  $- \otimes_B T$  give a one-one correspondence between the  $T$ -resolving subcategories of  $T^\perp$  and the resolving subcategories of  ${}^\perp(D_B T)$ .*

**Proof** Let  $\mathcal{U}$  be a  $T$ -resolving subcategory of  $T^\perp$ . Then  $\text{add } B = \text{add}(\text{Hom}_A(T, T)) \subseteq \text{Hom}_A(T, \mathcal{U})$ , and  $\text{Hom}_A(T, \mathcal{U})$  contains all indecomposable projective left  $B$ -modules.

Suppose  $0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$  is an exact sequence of  $B$ -modules with  $V_1, V_3 \in \text{Hom}_A(T, \mathcal{U}) \subseteq {}^\perp(D_B T)$ , then  $V_2 \in {}^\perp(D_B T)$  since  ${}^\perp(D_B T)$  is closed under extension. According to Lemma 2.6, there exist  $U_1, U_3 \in \mathcal{U}$ ,  $U_2 \in T^\perp$  such that  $V_i = \text{Hom}_A(T, U_i)$ ,  $U_i = V_i \otimes_B T$  ( $i = 1, 2, 3$ ). Since  $D\text{Tor}_1(V_3, T) \cong \text{Ext}_B^1(V_3, D_B T) = 0$ , applying the functor  $- \otimes_B T$  to the exact sequence  $0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$ , we obtain an exact sequence  $0 \rightarrow V_1 \otimes_B T \rightarrow V_2 \otimes_B T \rightarrow V_3 \otimes_B T \rightarrow 0$ , hence we have an exact sequence  $0 \rightarrow U_1 \rightarrow U_2 \rightarrow U_3 \rightarrow 0$  in  $T^\perp$ . Since  $\mathcal{U}$  is  $T$ -resolving in  $T^\perp$ ,  $U_1$  and  $U_3 \in \mathcal{U}$ , we know that  $U_2 \in \mathcal{U}$ , and  $V_2 = \text{Hom}_A(T, U_2) \in \text{Hom}_A(T, \mathcal{U})$ , i.e.,  $\text{Hom}_A(T, \mathcal{U})$  is closed under extension.



Similarly, one can prove that  $\text{Hom}_A(T, \mathcal{U})$  is closed under the kernel of epimorphism. Hence  $\text{Hom}_A(T, \mathcal{U})$  is a resolving subcategory of  ${}^\perp(D_B T)$ .

On the other hand, by using the same method, we can prove that if  $\mathcal{V}$  is a resolving subcategory of  ${}^\perp(D_B T)$ , then  $\mathcal{V} \otimes_B T$  is a  $T$ -resolving subcategory in  $T^\perp$ . The proof is completed.  $\square$

Summarizing Lemma 4.1 and Lemma 4.2, we have the following.

**Theorem 4.3** *Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A T$ . Then there is a one-one correspondence between the  $T$ -contravariantly finite  $T$ -resolving subcategories of  $T^\perp$  and the contravariantly finite resolving subcategories in  ${}^\perp(D_B T)$ .*

Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A T$ . According to Lemma 2.7, there is a one-one correspondence between the equivalence classes of basic tilting  $B$ -modules and contravariantly finite subcategories of  $\text{mod-}B$ . Since  ${}^\perp(D_B T)$  is a resolving and contravariantly finite subcategory of  $\text{mod-}B$ , restricting the above one-one correspondence on the subcategory  ${}^\perp(D_B T)$  also yields the following lemma.

**Lemma 4.4** *Let  $T$  be a tilting  $A$ -module and  $B = \text{End}_A T$ . Let  $M \in {}^\perp(D_B T)$ . Then  $M \rightarrow \widetilde{\text{add} M}$  and  $\mathcal{X} \rightarrow \mathcal{X} \cap \mathcal{X}^\perp$  give a one-one correspondence between the equivalence classes of basic tilting  $B$ -modules in  ${}^\perp(D_B T)$  and the resolving contravariantly finite subcategories belonging to  $\mathcal{X} \subseteq \mathcal{P}^{<\infty}(\text{mod} B) \cap {}^\perp(D_B T)$ .*

**Lemma 4.5** *Let  $T$  be a tilting  $A$ -module,  $B = \text{End}_A T$ , and  $\mathcal{X}$  be a subcategory of  ${}^\perp(D_B T)$ . Then we have following:*

- (1)  $\check{\mathcal{X}} \otimes_B T = \widetilde{\mathcal{X} \otimes_B T} \cap T^\perp$ .
- (2)  $(\mathcal{X}^\perp \cap {}^\perp(D_B T)) \otimes_B T = (\mathcal{X} \otimes_B T)^\perp \cap T^\perp$ .

**Proof** (1) Let  $M \otimes_B T \in \check{\mathcal{X}} \otimes_B T$  with  $M \in \check{\mathcal{X}}$ . Then there is an exact sequence:

$$0 \rightarrow M \rightarrow X_0 \xrightarrow{f_0} X_1 \cdots \xrightarrow{f_{n-1}} X_n \rightarrow 0 \text{ with } X_i \in \mathcal{X}.$$

Let  $K_i = \ker f_i$  and  $M = \ker f_0 = K_0$ . Since  ${}^\perp(D_B T)$  is resolving in  $\text{mod-}B$

and  $\mathcal{X} \subseteq {}^\perp(D_B T)$ , we have that  $K_i \in {}^\perp(D_B T)$  and  $M = K_0 \in {}^\perp(D_B T)$ , hence  $M \otimes_B T \in T^\perp$ . Note that  $D\mathrm{Tor}_1(K_i, {}_B T) \cong \mathrm{Ext}_B^1(K_i, D_B T) = 0$ , applying the functor  $- \otimes_B T$  on the above sequence, we obtain an exact sequence:

$$0 \rightarrow M \otimes_B T \rightarrow X_0 \otimes_B T \rightarrow X_1 \otimes_B T \cdots \rightarrow X_n \otimes_B T \rightarrow 0,$$

it follows that  $M \otimes_B T \in \widetilde{\mathcal{X} \otimes_B T}$ , hence  $M \otimes_B T \in \widetilde{\mathcal{X} \otimes_B T} \cap T^\perp$ .

On the other hand, for any  $N \in \widetilde{\mathcal{X} \otimes_B T} \cap T^\perp$ , there exists an exact sequence

$$0 \rightarrow N \rightarrow Y_0 \otimes_B T \rightarrow Y_1 \otimes_B T \cdots \rightarrow Y_m \otimes_B T \rightarrow 0$$

with  $Y_i \in \mathcal{X}$ . Applying the functor  $\mathrm{Hom}_A(T, -)$ , by using Lemma 2.6 and  $N \in T^\perp$ , we have an exact sequence  $0 \rightarrow \mathrm{Hom}_A(T, N) \rightarrow Y_0 \rightarrow Y_1 \cdots \rightarrow Y_m \rightarrow 0$ . Hence  $\mathrm{Hom}_A(T, N) \in \check{\mathcal{X}}$ , and  $N = \mathrm{Hom}_A(T, N) \otimes_B T \in \check{\mathcal{X}} \otimes_B T$ .

(2) Let  $M \otimes_B T \in (\mathcal{X}^\perp \cap {}^\perp(D_B T)) \otimes_B T$  with  $M \in \mathcal{X}^\perp \cap {}^\perp(D_B T)$ . According to Lemma 2.5 (2), for any  $X \otimes_B T \in \mathcal{X} \otimes_B T$  with  $X \in \mathcal{X}$ , we have  $\mathrm{Ext}_A^i(X \otimes_B T, M \otimes_B T) \cong \mathrm{Ext}_B^i(X, M) = 0$ , hence  $M \otimes_B T \in (\mathcal{X} \otimes_B T)^\perp$ . Note that  $M \in {}^\perp(D_B T)$ , thus  $M \otimes_B T \in T^\perp$ , hence  $M \otimes_B T \in (\mathcal{X} \otimes_B T)^\perp \cap T^\perp$ .

On the other hand, let  $N \in (\mathcal{X} \otimes_B T)^\perp \cap T^\perp$ . Then  $\mathrm{Hom}_A(T, N) \in {}^\perp(D_B T)$ . For every  $X \in \mathcal{X}$ , by using Lemma 2.5 (2) again, we have

$$\mathrm{Ext}_B^i(X, \mathrm{Hom}_A(T, N)) \simeq \mathrm{Ext}_A^i(X \otimes_B T, \mathrm{Hom}_A(T, N) \otimes_B T) \simeq \mathrm{Ext}_A^i(X \otimes_B T, N) = 0.$$

It follows that  $\mathrm{Hom}_A(T, N) \in \mathcal{X}^\perp$  and  $\mathrm{Hom}_A(T, N) \in \mathcal{X}^\perp \cap {}^\perp(D_B T)$ , Hence  $N = \mathrm{Hom}_A(T, N) \otimes_B T \in (\mathcal{X}^\perp \cap {}^\perp(D_B T)) \otimes_B T$ . The proof is completed.  $\square$

Let  $T$  be a tilting  $A$ -module, and let  $L$  and  $L'$  are  $T$ -tilting modules. We say  $L$  and  $L'$  are equivalent if  $\mathrm{add} L = \mathrm{add} L'$ .

**Theorem 4.6** *Let  $T$  be a tilting  $A$ -module. Then for any  $L \in T^\perp$ ,  $\mathrm{add} L \rightarrow \widetilde{\mathrm{add} L} \cap T^\perp$  and  $\mathcal{U} \rightarrow \mathcal{U} \cap \mathcal{U}^\perp$  give a one-one correspondence between equivalence class of basic  $T$ -tilting  $A$ -modules in  $T^\perp$  and the  $T$ -contravariantly finite  $T$ -resolving subcategories  $\mathcal{U}$  of  $T^\perp$  which is contained in  $\mathcal{T}^{<\infty}(T)$ .*

**Proof** For any  $X \in T^\perp$  and  $Y \in {}^\perp(D_B T)$ , it is easy to see that  $\text{Hom}_A(T, \text{add}X) = \text{add}\text{Hom}_A(T, X)$ ,  $(\text{add}Y) \otimes_B T = \text{add}(Y \otimes_B T)$ . According to Lemma 3.1 and Lemma 2.6, we have  $\text{T-pd}X = \text{pd}_B(\text{Hom}_A(T, X))$  and  $\text{Hom}_A(T, \mathcal{T}^{<\infty}(T)) = \mathcal{P}^{<\infty}(\text{mod}B) \cap {}^\perp(D_B T)$ .

(1) Let  $L$  be a basic  $T$ -tilting  $A$ -module in  $T^\perp$ . Then by Theorem 3.2,  $Y = \text{Hom}_A(T, L) \in {}^\perp(D_B T)$  is a basic tilting  $B$ -module. By Lemma 4.4, we know that  $\widetilde{\text{add}Y}$  is contravariantly finite in  $\text{mod-}B$ , which is contained in  $\mathcal{P}^{<\infty}(\text{mod}B) \cap {}^\perp(D_B T)$ . Applying the functor  $-\otimes_B T$  on  $\widetilde{\text{add}Y}$ , and using Lemma 4.4 and Lemma 4.5, we know that  $\widetilde{\text{add}Y} \otimes_B T = \widetilde{\text{add}L} \cap T^\perp$  is a  $T$ -contravariantly finite  $T$ -resolving subcategory of  $T^\perp$  which is contained in  $\mathcal{T}^{<\infty}(T)$ .

(2) Let  $\mathcal{U}$  be a  $T$ -contravariantly finite  $T$ -resolving subcategory of  $T^\perp$  and  $\mathcal{U} \subseteq \mathcal{T}^{<\infty}(T)$ . Then  $\text{Hom}_A(T, \mathcal{U}) \subseteq \mathcal{P}^{<\infty}(\text{mod}B) \cap {}^\perp(D_B T)$ . According to Lemma 4.1 and Lemma 4.2, we know that  $\text{Hom}_A(T, \mathcal{U})$  is a contravariantly finite resolving subcategory, which is contained in  ${}^\perp(D_B T)$ . By using Lemma 4.4, there exists a tilting  $B$ -module  $M \in {}^\perp(D_B T)$  such that  $\text{Hom}_A(T, \mathcal{U}) \cap (\text{Hom}_A(T, \mathcal{U}))^\perp = \text{add}M$ , hence  $\text{Hom}_A(T, \mathcal{U}) \cap ((\text{Hom}_A(T, \mathcal{U}))^\perp \cap {}^\perp(D_B T)) = \text{add}M$ . Applying the functor  $-\otimes_B T$  and using Lemma 4.5, we have  $\mathcal{U} \cap (\mathcal{U}^\perp \cap T^\perp) = \text{add}(M \otimes_B T)$ , thus  $\mathcal{U} \cap \mathcal{U}^\perp = \text{add}(M \otimes_B T)$ , since  $\mathcal{U}$  is a subcategory of  $T^\perp$ . According to Theorem 3.2,  $M \otimes_B T$  is a  $T$ -tilting  $A$ -module in  $T^\perp$ .

(3) Now we show that  $\text{add}L \rightarrow \widetilde{\text{add}L} \cap T^\perp$  and  $\mathcal{U} \rightarrow \mathcal{U} \cap \mathcal{U}^\perp$  are a pair of converse functors.

Let  $L$  be a basic  $T$ -tilting  $A$ -module. Then  $Y = \text{Hom}_A(T, L) \in {}^\perp(D_B T)$  is a basic tilting  $B$ -module. According to Lemma 4.4, we have that  $\widetilde{\text{add}Y} \cap (\widetilde{\text{add}Y})^\perp = \text{add}Y$  and  $\text{add}Y = \widetilde{\text{add}Y} \cap (\widetilde{\text{add}Y})^\perp \cap {}^\perp(D_B T)$ . Applying the the functor  $-\otimes_B T$  and using Lemma 4.5, we have

$$\text{add}L = (\widetilde{\text{add}L} \cap T^\perp) \cap (\widetilde{\text{add}L} \cap T^\perp)^\perp \cap T^\perp = (\widetilde{\text{add}L} \cap T^\perp) \cap (\widetilde{\text{add}L} \cap T^\perp)^\perp.$$

On the other hand, let  $\mathcal{U}$  be a  $T$ -contravariantly finite  $T$ -resolving subcategory and  $\mathcal{U} \subseteq \mathcal{T}^{<\infty}(T)$ . By Lemma 4.2,  $\mathcal{V} = \text{Hom}_A(T, \mathcal{U})$  is a contravariantly finite resolving subcategory, which is contained in  $\mathcal{P}^{<\infty}(\text{mod}B) \cap {}^\perp(D_B T)$ . By using Lemma 4.4, we have  $\widetilde{\mathcal{V}} \cap \mathcal{V}^\perp = \mathcal{V}$ . Applying the functor  $-\otimes_B T$  on the both side

and using Lemma 4.5, we have  $\mathcal{U} = \widetilde{\mathcal{U} \cap \mathcal{U}^\perp} \cap T^\perp$ . The proof is finished.  $\square$

Combining Theorem 3.5 and Theorem 4.6, we obtain the following result, which seems to have independent interests.

**Theorem 4.7** *Let  $A$  be either a 1-Gorenstein algebra or a  $m$ -replicated algebra over a hereditary algebra, and  $T$  be a tilting  $A$ -module. Then there is a one-one correspondence between the equivalence classes of basic tilting  $A$ -modules in  $T^\perp$  and the  $T$ -contravariantly finite  $T$ -resolving subcategories contained in  $\mathcal{T}^{<\infty}(T)$ .*

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